ON THE MEHLER-FOCK TRANSFORM
OF GENERALIZED FUNCTIONS

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Abstract
In this paper the Mehler-Fock transform with kernel the associated Legendre function $P_{\frac{n}{2}+\tau_1}(t)$ ($\mu$ being a complex parameter and $\tau$ a positive real number) is defined on certain space of generalized functions and its inversion formula is established.

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1 Introduction.

The integral transform

$$\int_1^{\infty} f(t) P_{\frac{n}{2}+\tau_1}(t) dt \quad (\tau \text{ real } > 0) \quad (1.1)$$

was introduced by Mehler (1881) and an inversion formula proved by Fock (1943). Usually it is called Mehler-Fock transform. Several authors (Rosenthal [12], Nasim [10], Braaksma-Meulenbeld [1]) have developed some generalizations by including in the kernel an associated Legendre function $P_{\frac{n}{2}+\tau_1}(t)$, $\mu$ being a complex parameter. The extension of the transform (1.1) to spaces of generalized functions has been investigated by Tiwari-Pandey [14]. Glaeske-Hess ([4] and [5]) introduced a convolution for the transform studied by Buggle in his thesis and whose kernel is the function $P_{\frac{n}{2}+\tau_1}(t) (n = 0, 1, 2, \ldots)$.

The purpose of this paper is to extend the Mehler-Fock transform with kernel $P_{\frac{n}{2}+\tau_1}(t)$, ($\mu \in \mathbb{C}$, $\tau \in \mathbb{R}_+$) to certain space of generalized functions.

According to Zemanian [16], we introduce the testing function space $M_{a,\mu}$ ($a \in [0, 1]$) and $\mu \in \mathbb{C}$) which contains the kernel $P_{\frac{n}{2}+\tau_1}(t)$. As usual, $M_{a,\mu}^*$ denotes the dual space of $M_{a,\mu}$.

The Mehler-Fock transform of $f \in M_{a,\mu}$ is defined by:

$$F(\tau) = < f(t), P_{\frac{n}{2}+\tau_1}(t) > \quad \tau \in \mathbb{R}_+ \quad (1.2)$$


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An inversion formula for the generalized transformation \((1.2)\) on the space \(\mathcal{E}'(I)\) is proved.

In the sequel \(I\) denotes the real interval \((1,\infty)\), \(\mathbb{R}_+\) the set of the positive real numbers and \(\mathbb{N}_0\) the set of non-negative integers. \(\mathcal{D}(I)\) and \(\mathcal{E}(I)\) are the well-known Schwartz spaces of testing functions on the open interval \(I\). \(\mathcal{D}'(I), \mathcal{E}'(I)\) are their duals \([13]\).

## 2 The testing function space and its dual.

Let \(a \in [0, \frac{1}{2})\) and \(\mu \in \mathbb{C}\). \(M_{a,\mu}\) is the space of all infinitely differentiable complex-valued functions \(\phi(t)\) defined on \(I\) such that

\[
\gamma_{k,a,\mu}(\phi) = \sup_{t \in I} |t^k A_t^a \phi(t)|
\]

exists for every nonnegative integer \(k\), where \(A_t\) is the differential operator

\[
A_t = (t^2 - 1)^{\frac{\mu}{2}} D_t(t^2 - 1)^{\mu + \frac{1}{2}} D_t(t^2 - 1)^{-\frac{\mu}{2}}.
\]

We equip \(M_{a,\mu}\) with the topology arising from the family of seminorms \(\{\gamma_{k,a,\mu}\}\) of which \(\gamma_{0,a,\mu}\) is a norm. Thus, \(M_{a,\mu}\) is a countably multinormed, locally convex, Hausdorff space. By a well-known technique (Zemanian \([16]\)), it follows immediately that \(M_{a,\mu}\) is sequentially complete, i.e., a Fréchet space.

In the next lemma we establish that \(D^m_{\pm} P^{-\mu}_{-\frac{1}{2}+\imath}(t)\) belongs to \(M_{a,\mu}\) for all \(m \in \mathbb{N}_0\).

### Lemma 2.1

For \(R > -\frac{1}{2}\), \(a \in [0, \frac{1}{2})\) and \(k, m \in \mathbb{N}_0\), there exists a constant \(C(a, k, m, \mu)\) such that

\[
\gamma_{k,a,\mu} \left( D^m_{\pm} P^{-\mu}_{-\frac{1}{2}+\imath}(t) \right) \leq C(a, k, m, \mu) \left( \mu + \frac{1}{2} \right)^k + \tau^k
\]

with \(C\) independent of \(\tau\).

**Proof.** Starting from the integral representation ([2] 3.7(6)):

\[
P^{-\mu}_{-\frac{1}{2}+\imath}(t) = \frac{2^{-\mu} (t^2 - 1)^{-\frac{\mu}{2}}}{\sqrt{\pi} (\mu + \frac{1}{2})} \int_0^\infty \left[ t + (t^2 - 1)^\frac{1}{2} \cos \xi \right]^{-\mu-\frac{1}{2}+\imath \tau} (\sin \xi)^m d\xi
\]

valid for \(R > -\frac{1}{2}\), one has:

\[
D^m_{\pm} P^{-\mu}_{-\frac{1}{2}+\imath}(t) = \frac{2^{-\mu} (t^2 - 1)^{-\frac{\mu}{2}}}{\sqrt{\pi} (\mu + \frac{1}{2})} \int_0^\infty \left[ t + (t^2 - 1)^\frac{1}{2} \cos \xi \right]^{-\mu-\frac{1}{2}+\imath \tau} (-i)^m \left[ \ln \left( t + (t^2 - 1)^\frac{1}{2} \cos \xi \right) \right]^m (\sin \xi)^{2\mu} d\xi.
\]

Thus,

\[
|D^m_{\pm} P^{-\mu}_{-\frac{1}{2}+\imath}(t)| \leq M \left[ \ln \left( t + (t^2 - 1)^\frac{1}{2} \right) \right]^m P^{-R\mu}_{-\frac{1}{2}}(t).
\]

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Now, by the asymptotic behavior of $P_{-\frac{1}{4}+\epsilon}^\mu(t)$ (cf. [11] 12.20) it follows that:

$$\gamma_{a,\nu} \left( D_{\eta}^{-\mu} P_{-\frac{1}{4}+\epsilon}^\mu(t) \right) \leq C(a, m)$$ (2.4)

for a certain $C(a, m) > 0$. Finally, since $P_{-\frac{1}{4}+\epsilon}^\mu(t)$ satisfies the relation:

$$A_{\nu} P_{-\frac{1}{4}+\epsilon}^\mu(t) = -\left( \mu + \frac{1}{2} \right)^2 + r^2 P_{-\frac{1}{4}+\epsilon}^\mu(t)$$ (2.5)

we conclude (2.2) by a straightforward verification.

The following propositions summarize some further properties of the space $M_{a,\mu}$ and its dual.

**Proposition 2.1** For $0 \leq a < b < 1/2$, $D(I) \subset M_{a,\mu} \subset E(I)$, all the inclusions being continuous. Moreover, $M_{a,\mu}$ is a dense subspace of $E(I)$ but $D(I)$ is not dense in $M_{a,\mu}$.

The proof is based on a technique to be found in [16], Chap. 9.3.

**Proposition 2.2** The dual $M'_{a,\mu}$ endowed with the usual weak topology is a Hausdorff locally convex, sequentially complete space of generalized functions.

For $f \in M'_{a,\mu}$ there exists $C > 0$ and $r \in N_0$ ($C$ and $r$ depending on $f$) such that

$$| < f, \phi > | \leq C \max_{0 \leq k \leq r} \gamma_{k,a,\mu}(\phi)$$ (2.6)

**Proof.** This result is an immediate consequence from [16] (theorems 1.8-1 and 1.8-3).

**Proposition 2.3** A locally integrable function $f$ in $I$ such that $t^{-a} f(t)$ is absolutely integrable on $I$, gives rise to a regular generalized function on $M_{a,\mu}$ through

$$< f, \phi > = \int_{\lambda}^{\infty} f(t) \phi(t) dt \quad \forall \phi \in M_{a,\mu}$$

**Proof.** That $f$ is truly a member of $M_{a,\mu}$ follows from the inequality:

$$| < f, \phi > | = \left| \int_{\lambda}^{\infty} f(t) \phi(t) dt \right| \leq \gamma_{a,\mu}(\phi) \int_{\lambda}^{\infty} |t^{-a} f(t)| dt$$

and the hypotheses.

**Proposition 2.4** The differential operator $A_{\mu}$ is a continuous linear mapping from $M_{a,\mu}$ into itself. $A_{\nu}$, the adjoint of $A_{\mu}$, maps continuously $M'_{a,\mu}$ into itself.

**Proof.** Observe that $\gamma_{k,a,\mu}(A_{\mu} \phi(t)) = \gamma_{k+1,a,\mu}(\phi(t))$.  

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3 The generalized transform.

For \( f \in \mathcal{M}_{a,n} \), the generalized Mehler-Fock transform is defined by

\[
MF[f] = F(\tau) = \langle f(t), P_{-\frac{1}{2}+it}^{-\mu}(t) \rangle \quad \tau \in \mathbb{R}_+
\]  

(3.1)

Note that definition (3.1) has sense since \( P_{-\frac{1}{2}+it}^{-\mu}(t) \in \mathcal{M}_{a,n} \) by taking \( m = 0 \) in Lemma 2.1.

Proposition 3.1 For all \( f \in \mathcal{M}_{a,n} \) and \( k \in \mathbb{N}_0 \) one has:

\[
MF[A_k^+ f] = (-1)^k \left[ \left( \mu + \frac{1}{2} \right)^2 + \tau^2 \right]^k MF[f]
\]  

(3.2)

**Proof.** It follows immediately from (2.5).

The next two Theorems establish the kind of functions that is obtained by means of (3.1).

Theorem 3.1 The generalized transform \( F(\tau) \) of \( f \in \mathcal{M}_{a,n} \) with \( R_{a\mu} > -\frac{1}{2} \), is a smooth function in \( \mathbb{R} \) and

\[
D^{m}_\tau F(\tau) = \langle f(t), D^{m}_\tau P_{-\frac{1}{2}+it}^{-\mu}(t) \rangle \quad m \in \mathbb{N}_0
\]  

(3.3)

**Proof.** By Lemma 2.1 it follows that \( D^{m}_\tau P_{-\frac{1}{2}+it}^{-\mu}(t) \in \mathcal{M}_{a,n} \). Thus, we need merely to prove (3.3), what we do through an inductive argument. Assume that (3.3) is true for \( m \) replaced by \( m - 1 \). It is true by definition for \( m = 0 \). Letting \( \tau \) be fixed and \( \Delta \tau \neq 0 \), consider

\[
\frac{1}{\Delta \tau} \left[ D^{m-1}_\tau F(\tau + \Delta \tau) - D^{m-1}_\tau F(\tau) \right] = \langle f(t), D^{m}_\tau P_{-\frac{1}{2}+it}^{-\mu}(t) \rangle = \langle f(t), \Delta F(\tau) \rangle
\]  

(3.4)

where

\[
\Delta F(\tau) = \frac{1}{\Delta \tau} \left[ D^{m-1}_\tau P_{-\frac{1}{2}+i(\tau+\Delta \tau)}^{-\mu}(t) - D^{m-1}_\tau P_{-\frac{1}{2}+it}^{-\mu}(t) \right] - D^{m}_\tau P_{-\frac{1}{2}+it}^{-\mu}(t).
\]

For any non negative integer \( k \) we may write:

\[
A_k^+ \Delta F(\tau) = \frac{(-1)^k}{\Delta \tau} \int_{\tau}^{\tau + \Delta \tau} dx \int_{\mathbb{R}} D^{m+1}_\eta \left[ \left( \mu + \frac{1}{2} \right)^2 + \eta^2 \right]^k P_{-\frac{1}{2}+i\eta}(t) \eta.
\]

Next, let \( \Lambda \) denote the interval \( \tau - |\Delta \tau| < \eta < \tau + |\Delta \tau| \). Then,

\[
|t^* A_k^+ \Delta F(\tau)| \leq \frac{|\Delta \tau|}{2} t^* \sup_{\eta \in \Lambda} \left| D^{m+1}_\eta \left[ \left( \mu + \frac{1}{2} \right)^2 + \eta^2 \right]^k P_{-\frac{1}{2}+i\eta}(t) \right|.
\]  

(3.5)

Now, by (2.3),

\[
t^* \sup_{\eta \in \Lambda} \left| D^{m+1}_\eta \left[ \left( \mu + \frac{1}{2} \right)^2 + \eta^2 \right]^k P_{-\frac{1}{2}+i\eta}(t) \right|
\]
is bounded on } 1 < t < \infty \text{ taking } |\Delta \tau| < 1.

Therefore, it follows from (3.5) that } I_{\Delta \tau}(t) \text{ converges in } M_{\alpha, \mu} \text{ to zero as } \Delta \tau \to 0.

Since } f \in M_{\alpha, \mu}, (3.4) \text{ converges to zero as } \Delta \tau \to 0. \text{ This completes our inductive proof of (3.3).}

**Theorem 3.2**

Let } F(\tau) = Mf[|f|, \text{ with } f \in M_{\alpha, \mu}. \text{ If } R_\mu > -\frac{1}{2} \text{ then, one has:}

i) For all } m \in \mathbb{N}_0, F^{(m)}(\tau) = O(1) \text{ for } \tau \to 0^+.

ii) There exists an } \tau_0 \in \mathbb{N}_0 \text{ such that } F(\tau) = O(e^{\tau^2}), \tau \to \infty.

**Proof.** Part i) follows immediately from Theorem 3.1, (2.6) and (2.2). To prove ii) we recall that } P_{-\frac{1}{2}+\tau}(t) = O(1) \text{ as } |t| \to \infty, \text{ for } R_\mu > -\frac{1}{2}, \text{ (cf. [10])}. \text{ We get ii) by combining this fact with (2.5) and (2.6).}

## 4 Generalized inversion formula.

In order to establish the generalized inversion formula it is needed to recall the definition of the } M_{c, \gamma}^{-1}(L) \text{ spaces introduced in [15]:}

Let } c, \gamma \in \mathbb{R} \text{ with } 2 \text{ sgn } c + \text{ sgn } \gamma \geq 0. \text{ The space of functions } f(x) \text{ which can be represented in the form of}

\[
f(x) = \frac{1}{2\pi i} \int \rho(s)x^{-s}ds \quad x \in (0, \infty)
\]

where } \sigma = \{s \in \mathbb{C} : Re_s = \frac{1}{2}\}, \rho(s) = s^{-\sigma}e^{-\epsilon|\text{Im} s|}; |F(s)|ds < \infty, \text{ is denoted by } M_{c, \gamma}^{-1}(L). \text{ Note that } D(1) \subset M_{0, \gamma}^{-1}(L), \text{ for all } n \in \mathbb{N}_0.

**Lemma 4.1**

Let } f(t) \text{ be such that } g(t) = e^{-\mu(t+1)}f(2t+1) \in M_{c, \gamma}^{-1}(L) \text{ with } 2 \text{ sgn}(c+1) + \text{ sgn } (\gamma - R_\mu + 1) \geq 0. \text{ Then, there exists the integral}

\[
F(\tau) = \frac{1}{2\pi i} \int \Gamma(\mu + \frac{1}{2} + \tau)\Gamma(\mu + \frac{1}{2} - \tau)\Gamma(\frac{1}{2} + \frac{1}{2} - \alpha + ir - s)\Gamma(\frac{1}{2} + \frac{1}{2} - \alpha - ir - s)\Gamma(\frac{1}{2} + \alpha + s)g^\ast(1-s)ds
\]

where } g^\ast \text{ is the Mellin transform of } g. \text{ Moreover, if } R_\mu > 0, R_\mu(2\alpha - \mu) < 0, R_\mu(2\alpha + \mu) > -1, \text{ one has:}

\[
F(\tau) = \int_1^{\infty} P_{-\frac{1}{2}+\tau}(t)f(t)dt \quad (4.1)
\]

**Proof.** Since } g(t) \in M_{c, \gamma}^{-1}(L), \text{ the first part follows from the asymptotic behavior of the Gamma function. For the second one, observe that the integral in (4.1) exists by hypothesis. Moreover, it can be written}

\[
F(\tau) = 2 \int_0^{\infty} e^{-\mu(t+1)-\frac{1}{2}P_{-\frac{1}{2}+\tau}(2t+1)}g(t)dt
\]

and by the imposed conditions on } g, \text{ one has:

\[
F(\tau) = 2 \int_0^{\infty} e^{-\mu(t+1)-\frac{1}{2}P_{-\frac{1}{2}+\tau}(2t+1)}\frac{1}{2\pi i} \int \Gamma(z)g^\ast(1-s)ds.
\]

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The Fubini's theorem allows to change the order of the integration and the conclusion follows.

Lemma 4.2 Let \( \alpha, \mu \) and \( s \) be complex parameters with \( R_{\alpha} > 0 \), \( R_{\mu} > 0 \), and \( R_{s} = \frac{1}{2}, \frac{1}{4} < R_{\mu} - 2\alpha < \frac{1}{2} \). The following integral representation

\[
\left( \frac{z + 1}{x - 1} \right)^{-\frac{s}{2}} \int_{0}^{\infty} z^{\frac{s}{2} - \alpha - 1} C_{\mu} \left( \frac{x - 1}{2} \right) dz \int_{-\infty}^{\infty} e^{\theta \left( \mu - 2\alpha - 1 - 2s \right)} d\theta \int_{0}^{x} C_{0} \left( z e^{\theta} \right) \sin \pi \theta \alpha \sum_{n=0}^{\infty} \frac{(\pi \sqrt{2 z})^{2n}}{(2n)!} \]

holds, \( \Psi \) being \( 2 \text{chu} - 2 \text{ch} \beta \). Here, \( C_{\mu} \) denotes the Bessel-Clifford function of the first kind and order \( \mu \) (cf. [6]).

Remark 4.1 The function \( C_{\mu} \) of this lemma is related to the Bessel function \( J_{\mu} \) through \( C_{\mu}(x) = \frac{x^{\mu}}{\Gamma(\frac{1}{2})} J_{\mu}(2\sqrt{x}) \) (see also [6]).

Proof. Let us multiply both sides of the integral representation (see [8])

\[
\frac{2}{\pi} K_{2\mu}(2\sqrt{y}) K_{2\mu}(2\sqrt{y}) \text{sh} \pi \tau = \frac{2}{\sqrt{\pi}} \int_{\frac{1}{2} \log \frac{\pi}{2}}^{\infty} C_{0} \left( 2\sqrt{\chi y} \right) \sin \pi \theta \alpha \sum_{n=0}^{\infty} \frac{(\pi \sqrt{2 \chi y})^{2n}}{(2n)!} \]

by \( y^{\frac{s}{2} - \alpha - 1} \) with \( R_{s} = \frac{1}{2} \), integrate respect to \( y \) from 0 to \( \infty \) and multiply the result by \( \frac{y^{\frac{s}{2} - \alpha - 1}}{2} \). Taking into account that ([3] 10.2(1)):

\[
\int_{0}^{\infty} y^{\frac{s}{2} - \alpha - 1} y K_{2\mu}(2\sqrt{y}) dy = \frac{1}{2} \sqrt{\pi} \left( \frac{\mu + 1}{2} + \frac{1}{2} - \alpha + i\tau - s \right) \Gamma \left( \frac{\mu + 1}{2} + \frac{1}{2} - \alpha - i\tau - s \right)
\]

and that ([3] 8.13(4)):

\[
\int_{0}^{\infty} x^{\frac{s}{2} - \alpha - 1} K_{2\mu}(2\sqrt{y}) dy = \frac{\pi}{c h \pi \tau \left( \frac{x + 1}{x - 1} \right)^{\frac{s}{2}}} P_{-\frac{1}{2} + i\tau}(x),
\]

a new integration with respect to \( z \) from 0 to \( \infty \) with \( \frac{1}{2} \log \frac{\pi}{2} = \theta \), leads to desired result (4.2).

The existence of the integral (4.2) can be justified by means of the absolute convergence, making use of the asymptotic behavior of the Bessel-Clifford functions \( C_{\mu}(x) \) (cf. [6]) and the imposed conditions.

We now establish the following two lemmas that we shall need in the proof of the inversion formula.

Lemma 4.3 If \( M F[f] = F(\tau), \phi \in \mathcal{D}(\text{I}) \) and we set

\[
\varphi(\tau) = \frac{1}{\pi} \tau \text{sh} \pi \tau \Gamma(\mu + \frac{1}{2} + i\tau) \Gamma(\mu + \frac{1}{2} - i\tau) \int_{1}^{\infty} \phi(t) P_{-\frac{1}{2} + i\tau}(t) dt,
\]

then,

\[
\int_{0}^{\tau} \varphi(\tau) < f(x), P_{-\frac{1}{2} + i\tau}(x) > d\tau = \left( f(x), \int_{0}^{\tau} \varphi(\tau) P_{-\frac{1}{2} + i\tau}(x) d\tau \right)
\]

for any fixed real number \( T > 0 \).
PROOF. Our conclusion is obvious when \( \int_1^\infty \phi(t)P_{-\frac{\mu}{2}+i\nu}(t)dt = 0 \).

Let us assume that \( \int_1^\infty \phi(t)P_{-\frac{\mu}{2}+i\nu}(t)dt \neq 0 \). By the asymptotic behavior of \( P_{-\frac{\mu}{2}+i\nu}(t) \) one has that

\[
\Theta_T(x) = \int_0^T \phi(t)P_{-\frac{\mu}{2}+i\nu}(x)dt
\]

is in \( M_{a,\mu} \), and this will insure that the right hand of (4.5) has sense. Writing

\[
Q(x, n) = \frac{T}{n} \sum_{j=1}^n \phi \left( \frac{pT}{n} \right) P_{-\frac{\mu}{2}+i\nu}(x)
\]

it follows

\[
< f(x), Q(x, n) > = \frac{T}{n} \sum_{j=1}^n < f(x), P_{-\frac{\mu}{2}+i\nu}(x) >.
\]

Because of the continuity of the functions involved, (4.7) tends to

\[
\int_0^T \phi(t) < f(x), P_{-\frac{\mu}{2}+i\nu}(x) > dt \quad \text{as } n \to \infty.
\]

Next, note that both \( \Theta_T(x) \) and \( Q(x, n) \) are members of \( M_{a,\mu} \) and consider

\[
x^a A^k_n (\Theta_T(x) - Q(x, n)) = x^a \int_0^T \phi(t) A^k_n P_{-\frac{\mu}{2}+i\nu}(x)dt - x^a \int_0^T \frac{T}{n} \sum_{j=1}^n \phi \left( \frac{pT}{n} \right) A^k_n P_{-\frac{\mu}{2}+i\nu}(x)dt =
\]

\[
= x^a \int_0^T \phi(t) (-1)^k \left( \mu + \frac{1}{2} \right)^2 + \tau^2 \right] P_{-\frac{\mu}{2}+i\nu}(x)dt -
\]

\[
- x^a \frac{T}{n} \sum_{j=1}^n \phi \left( \frac{pT}{n} \right) A^k_n P_{-\frac{\mu}{2}+i\nu}(x)dt.
\]

By virtue of the asymptotic behavior of \( P_{-\frac{\mu}{2}+i\nu}(x) \),

\[
\left| x^a \left( \mu + \frac{1}{2} \right)^2 + \tau^2 \right] P_{-\frac{\mu}{2}+i\nu}(x)
\]

tends to zero uniformly on \( 0 \leq \tau \leq T \) as \( x \to \infty \).

Given \( \varepsilon > 0 \), there exists \( N > 0 \) such that if \( x > N \) and \( 0 \leq \tau \leq T \), (4.9) is bounded by

\[
\frac{\varepsilon}{3} \left| \int_0^T \phi(t)dt \right|^{-1}
\]

which is a finite quantity since \( \int_1^\infty \phi(t)P_{-\frac{\mu}{2}+i\nu}(t)dt \neq 0 \). Hence

\[
\sup_{x > N} |x^a A^k_n \Theta_T(x)| < \frac{\varepsilon}{3}.
\]

Also, for all \( n \),

\[
\sup_{x > N} |x^a A^k_n Q(x, n)| < \frac{\varepsilon}{3} \left| \int_0^T \phi(t)dt \right|^{-1} \frac{T}{n} \sum_{j=1}^n \left| \phi \left( \frac{pT}{n} \right) \right|.
\]

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Thus, there exists an \( n_0 \) such that, for \( n > n_0 \) and \( x > N \),

\[
|z^a A^+_\mu [\Theta_T(x) - Q(x, n)]| < \epsilon.
\]

Moreover, on the domain \( \{(x, \tau) : 1 \leq x \leq N, 0 \leq \tau \leq T\} \), \( P^{-\mu}_{-\frac{1}{2} + it} (x) \) is an uniformly continuous function of \( (x, \tau) \). Hence, there exists \( n_1 \in \mathbb{N}_0 \) such that for \( n > n_1 \),

\[
|z^a A^+_\mu [\Theta_T(x) - Q(x, n)]| < \epsilon
\]
on \( 1 < x < N \). Therefore, when \( n > \max(n_0, n_1) \),

\[
|z^a A^+_\mu [\Theta_T(x) - Q(x, n)]| < \epsilon
\]
on \( 1 < x < \infty \). This completes the proof.

**Lemma 4.4** Let \( \Theta_T(x) \) be given by (4.6), with \( \phi \in \mathcal{D}(I) \). If \( \alpha \) and \( \mu \) are complex parameters such that \( R_\alpha \alpha > 0, R_\mu \mu > 0 \), and \( \frac{1}{4} < R_\mu (\mu - 2\alpha) < \frac{1}{2}, R_\mu (2\alpha + \mu) > -1 \), then \( \Theta_T(x) \) converges in \( \mathcal{E}(I) \) to \( \phi(x) \) as \( T \to \infty \).

**Proof.** If the support of \( \phi \) is contained in the closed interval \( [c,d] \) with \( 1 < c < d < \infty \), one has:

\[
\Theta_T(x) = \frac{1}{\pi} \int_0^T \tau \text{sh} \pi \tau \Gamma(\mu + \frac{1}{2} + ir)\Gamma(\mu + \frac{1}{2} - ir)P^{-\mu}_{-\frac{1}{2} + it}(x)dt (4.10)
\]

By virtue of the smoothness of the functions and the finiteness of the limits of integration, we may repeatedly differentiate under the integral sign obtaining

\[
A^+_\mu \Theta_T(x) = \frac{1}{\pi} \int_0^T \tau \text{sh} \pi \tau \Gamma(\mu + \frac{1}{2} + ir)\Gamma(\mu + \frac{1}{2} - ir).
\]

\[
(-1)^k \left[ \left( \mu + \frac{1}{2} \right)^2 + \tau^2 \right]^k P^{-\mu}_{-\frac{1}{2} + it}(x)dr \int_0^d \phi(t)P^{-\mu}_{-\frac{1}{2} + it}(t)dt. (4.11)
\]

A partial integration leads to:

\[
A^+_\mu \Theta_T(x) = \frac{1}{\pi} \int_0^T \tau \text{sh} \pi \tau \Gamma(\mu + \frac{1}{2} + ir)\Gamma(\mu + \frac{1}{2} - ir)P^{-\mu}_{-\frac{1}{2} + it}(x)dr \int_0^d \left( A^+_\mu \phi \right)(t)P^{-\mu}_{-\frac{1}{2} + it}(t)dt. (4.12)
\]

By applying Lemma 4.1

\[
A^+_\mu \Theta_T(x) = \frac{2}{\pi} \int_0^T \tau \text{sh} \pi \tau P^{-\mu}_{-\frac{1}{2} + it}(x)dr. (4.13)
\]

where \( \phi_k(t) = t^{-\sigma}(t + 1)^k \left( A^+_\mu \phi \right)(2t + 1) \). \( \phi_k \) is the Mellin transform of \( \phi_k \), for every \( k \in \mathbb{N}_0 \) and \( \sigma = \{ s \in \mathbb{C} | R_\phi s = \frac{1}{2} \} \). By reversing the order of integrals by Fubini's theorem (which is permissible by the imposed conditions), and using Lemma 4.2 we obtain

\[
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\]
\[ A^k \Theta_T(x) = \frac{4}{\pi} \left( \frac{x + 1}{x - 1} \right)^{-\frac{\gamma}{2}} \frac{1}{2\pi i} \int_0^\infty \frac{\Gamma\left( \frac{\gamma}{2} + \alpha + s \right) - \phi^*(1-s)ds}{\Gamma(1 + \frac{\gamma}{2} - \alpha - s)} \phi^*(1-s)ds. \]

\[
\int_0^\infty x^\frac{\gamma}{2} - \alpha - s C_\mu \left( \frac{x - 1}{2} \right) dx \int_0^\infty e^{(\mu - 2\alpha + 1 - 2s)} d\theta \int_0^\infty C_\mu (xe^\theta \Psi) \left[ \int_0^T \tau \sin 2\tau u d\tau \right] du =
\]

\[
\frac{1}{\pi} \left( \frac{x + 1}{x - 1} \right)^{-\frac{\gamma}{2}} \frac{1}{2\pi i} \int_0^\infty \frac{\Gamma\left( \frac{\gamma}{2} + \alpha + s \right)}{\Gamma(1 + \frac{\gamma}{2} - \alpha - s)} \phi^*(1-s)(xe^z)^{-\alpha} \int_0^\infty x^\frac{\gamma}{2} - \alpha C_\mu \left( \frac{x - 1}{2} \right) dx.
\]

\[
\int_0^\infty e^{(\mu - 2\alpha + 1)} d\theta \int_0^\infty C_\mu (xe^\theta \Psi) \left( -\frac{\partial}{\partial u} \frac{\sin 2Tu}{u} \right) du. \tag{4.14}
\]

Observe that

\[
\frac{1}{\pi} \int_0^\infty \frac{\Gamma\left( \frac{\gamma}{2} + \alpha + s \right)}{\Gamma(1 + \frac{\gamma}{2} - \alpha - s)} \phi^*(1-s)ds \tag{4.15}
\]

represents the \( G^\mu_\phi \)-transform of \( \phi(t) \) evaluated at the point \( \zeta \) (see [15]). Let us denote (4.15) by \( (G^\mu_\phi)(\zeta) \). Now, by the asymptotic behavior of the \( C_\mu \) functions and the hypotheses, the order of integration can be changed in (4.14) to obtain:

\[
A^k \Theta_T(x) = \frac{1}{\pi} \left( \frac{x + 1}{x - 1} \right)^{-\frac{\gamma}{2}} \int_0^\infty \int_{-u}^u e^{(\mu - 2\alpha + 1)} d\theta.
\]

\[
\int_0^\infty (G^\mu_\phi)(xe^z)z^\frac{\gamma}{2} - \alpha C_\mu \left( \frac{x - 1}{2} \right) C_\mu (xe^z) \left( -\frac{\partial}{\partial u} \frac{\sin 2Tu}{u} \right) dz du. \tag{4.16}
\]

Next, by the change of variable \( xe^z = y \), (4.16) can be written as:

\[
A^k \Theta_T(x) = \frac{1}{\pi} \left( \frac{x + 1}{x - 1} \right)^{-\frac{\gamma}{2}} \int_0^\infty \int_{-u}^u e^{\theta} d\theta.
\]

\[
\int_0^\infty (G^\mu_\phi)(y)y^\frac{\gamma}{2} - \alpha C_\mu \left( \frac{x - 1}{2} \right) C_\mu (ye^{-\theta}) \left( -\frac{\partial}{\partial u} \frac{\sin 2Tu}{u} \right) dy du. \tag{4.17}
\]

Thus, integrating by parts and taking into account that

\[
\frac{\sin 2Tu}{u} \left( \frac{x + 1}{x - 1} \right)^{-\frac{\gamma}{2}} \int_{-u}^u e^{\theta} d\theta \int_0^\infty (G^\mu_\phi)(y)y^\frac{\gamma}{2} - \alpha C_\mu \left( \frac{x - 1}{2} \right) C_\mu (ye^{-\theta}) dz
\]

tends uniformly to zero for \( u \to 0 \) and \( u \to \infty \) if \( x \) belongs to any compact \( K \subset I \) and \( \frac{1}{2} < R_\mu(\mu - 2\alpha) < \frac{1}{2} \), one has that (4.17) is equal to:

\[
A^k \Theta_T(x) = \frac{1}{\pi} \left( \frac{x + 1}{x - 1} \right)^{-\frac{\gamma}{2}} \int_0^\infty \Phi(x, u) e^{\frac{2Tu}{u}} du \tag{4.18}
\]

where

\[
\Phi(x, u) =
\]

\[
e^{-u} \int_0^\infty (G^\mu_\phi)(y)y^\frac{\gamma}{2} - \alpha C_\mu \left( \frac{x - 1}{2} \right) ey^{2u} dy + e^u \int_0^\infty (G^\mu_\phi)(y)y^\frac{\gamma}{2} - \alpha C_\mu \left( \frac{x - 1}{2} \right) ey^{-2u} dy.
\]

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If $F(x, u)$ denotes the last term in (4.19), by the absolute convergence we can to differentiate under the integral sign, and making use of:

$$
\frac{\partial}{\partial u} C_0 \left(y e^{-2\theta}\right) = 2y e^{-2\theta}(e^{\theta} - 1)C_1 \left(y e^{-2\theta}\right) - \frac{\partial}{\partial \theta} C_0 \left(y e^{-2\theta}\right)
$$

we obtain:

$$
F(x, u) = 2e^{-u} \int_{-\infty}^{\infty} e^{-2\theta} d\theta \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{-2\theta}\right) C_1 \left(y e^{-2\theta}\right) dy -
$$

$$
-2 \int_{-\infty}^{\infty} e^{-2\theta} d\theta \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{-2\theta}\right) C_1 \left(y e^{-2\theta}\right) dy -
$$

$$
- \int_{-\infty}^{\infty} e^{-2\theta} d\theta \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{-2\theta}\right) \frac{\partial}{\partial \theta} C_0 \left(y e^{-2\theta}\right) dy.
$$

(4.20)

By virtue of the asymptotic behavior of the Bessel-Clifford functions we may interchange the order of integrals in the last term in (4.20). An integration by parts leads to:

$$
F(x, u) = e^{-u} \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{-2\theta}\right) dy -
$$

$$
- e^u \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{2\theta}\right) dy + F_1(x, u) - F_2(x, u) + F_3(x, u)
$$

(4.21)

where:

$$
F_1(x, u) = 2e^{-u} \int_{-\infty}^{\infty} e^{-2\theta} d\theta \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{-2\theta}\right) C_1 \left(y e^{-2\theta}\right) dy
$$

$$
F_2(x, u) = 2 \int_{-\infty}^{\infty} e^{-2\theta} d\theta \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{-2\theta}\right) C_1 \left(y e^{-2\theta}\right) dy
$$

$$
F_3(x, u) = \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} \int_{-\infty}^{\infty} \frac{d\theta}{C_0 \left(y e^{-2\theta}\right)}.
$$

The substitution of (4.21) in (4.19) yields:

$$
\Phi(x, u) = 2e^{u} \int_{0}^{\infty} (G\phi_k)(y) y^{\frac{\mu}{2} - \alpha} C_\mu \left(\frac{x - 1}{2} y e^{-2\theta}\right) dy + F_1(x, u) - F_2(x, u) + F_3(x, u)
$$

(4.22)

Next, the first integral in (4.22) is, by the conditions of the lemma, the Hankel-Clifford integral transform (cf. [7] and [9]) of $\phi_k$. By its inversion formula one has:
\[ \Phi(x,u) = 2 \left( \frac{x-1}{2} \right)^{-\frac{\alpha}{2}} e^{-\lambda u} \left( \frac{x-1}{2} e^{2u} + 1 \right)^{\frac{\alpha}{2}} (A^\alpha\phi) \left( (x-1)e^{2u} + 1 \right) + F_1(x,u) - F_2(x,u) + F_3(x,u). \]  

Consider now:

\[ A^\alpha (\Theta_T(x) - \phi(x)) = \]

\[ = \frac{2}{\pi} \left( \frac{x+1}{2} \right)^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda u} \left( \frac{x-1}{2} e^{2u} + 1 \right)^{\frac{\alpha}{2}} \left[ (A^\alpha\phi) \left( (x-1)e^{2u} + 1 \right) - A^\alpha\phi(x) \right] \frac{\sin 2Tu}{u} du + \]

\[ + \frac{1}{\pi} \int_0^\infty (F_1(x,u) - F_1(x,u) + F_3(x,u)) \frac{\sin 2Tu}{u} du. \]  

Assume that \( x \) belongs to a compact \( K \subset I \) and set:

\[ A^\alpha (\Theta_T(x) - \phi(x)) = \]

\[ = \left( \int_0^\delta + \int_\delta^\infty \right) v(x,u) \sin 2Tdu + \]

\[ + \frac{1}{\pi} \int_0^\infty (F_1(x,u) - F_2(x,u) + F_3(x,u)) \frac{\sin 2Tu}{u} du \]  

being

\[ v(x,u) = \frac{2}{\pi u} \left( \frac{x+1}{2} \right)^{-\frac{\alpha}{2}} e^{-\lambda u} \left( \frac{x-1}{2} e^{2u} + 1 \right)^{\frac{\alpha}{2}} \left[ (A^\alpha\phi) \left( (x-1)e^{2u} + 1 \right) - A^\alpha\phi(x) \right] \]

with \( \delta > 0 \). Let \( I_1, I_2, I_3 \) be the three integrals (in the same order) that appear in (4.25). By using the techniques of Zemanian [17] it can be proved that there exists \( T_1 > 0 \) such that \( |I_1| < \varepsilon, |I_2| < \varepsilon \) for \( T > T_1 \) and all \( x \in K \). Moreover, since the functions \( x^i C_0(x) \) and \( x^i C_1(x) \) are bounded (see [6]), and taking into account the conditions of the theorem and the estimation

\[ |(G\phi(x))(u)| < Cw^{-\frac{1}{2}} \]

where \( C \) is a suitable constant, it follows that

\[ \left( \frac{x+1}{x-1} \right)^{-\frac{\alpha}{2}} \frac{F_i(x,u)}{u} \in L(0,\infty), \quad i = 1,2,3 \]

and by the Riemann lemma we can conclude that

\[ \left( \frac{x+1}{x-1} \right)^{-\frac{\alpha}{2}} \int_0^\infty F_i(x,u) \frac{\sin 2Tu}{u} du \to 0 \]

uniformly in \( K \) as \( T \to \infty, i = 1,2,3 \). Thus, \( \Theta_T(x) \to \phi(x) \) in \( C(I) \) and the lemma is proven.

Our main result is the following inversion theorem:
Theorem 4.1 Let $f \in \mathcal{E}'(I)$ and set

$$F(\tau) = \langle f(t), P_{-\frac{\mu}{1+i\tau}}(t) \rangle.$$ 

If $R_\alpha > 0$, $R_\alpha > 0$, and $\frac{1}{4} < R_\alpha(\mu - 2\alpha) < \frac{1}{2}$, $R_\alpha(2\alpha + \mu) > -1$, then

$$\lim_{T \to \infty} \left( \frac{1}{\pi} \int_0^T \tau \sin \pi \tau \Gamma(\mu + \frac{1}{2} + i\tau)\Gamma(\mu + \frac{1}{2} - i\tau)P_{-\frac{\mu}{1+i\tau}}(t)F(\tau)dt, \phi(t) \right)$$

(4.26)

for all $\phi \in \mathcal{D}(I)$.

Proof. Let $\phi \in \mathcal{D}(I)$. We shall show that

$$\left( \frac{1}{\pi} \int_0^T \tau \sin \pi \tau \Gamma(\mu + \frac{1}{2} + i\tau)\Gamma(\mu + \frac{1}{2} - i\tau)P_{-\frac{\mu}{1+i\tau}}(t)F(\tau)dt, \phi(t) \right)$$

(4.27)

tends to $\langle f, \phi \rangle$ as $T \to \infty$. From the smoothness of $F(\tau)$ and the fact that the support of $\phi(t)$ is a compact subset of $I$, it follows that (4.27) is really a repeated integral in $(t, \tau)$, having a continuous integrand on a compact domain of integration. Thus, we can change the order of integration to obtain that (4.27) coincides with

$$\int_0^T \frac{1}{\pi} \tau \sin \pi \tau \Gamma(\mu + \frac{1}{2} + i\tau)\Gamma(\mu + \frac{1}{2} - i\tau) < f(x), P_{-\frac{\mu}{1+i\tau}}(x) > \int_0^\infty \phi(t)P_{-\frac{\mu}{1+i\tau}}(t)dtdr.$$

Moreover, by Lemma 4.3 this one is equal to

$$\left( f(x), \int_0^T \frac{1}{\pi} \tau \sin \pi \tau \Gamma(\mu + \frac{1}{2} + i\tau)\Gamma(\mu + \frac{1}{2} - i\tau) \int_0^\infty \phi(t)P_{-\frac{\mu}{1+i\tau}}(t)dtP_{-\frac{\mu}{1+i\tau}}(x)dr \right).$$

(4.28)

Then, $f \in \mathcal{E}'(I)$, and according to Lemma 4.4, the testing function inside (4.28) converges in $\mathcal{E}(I)$ to $\phi(x)$ as $T \to \infty$. Hence, (4.28) tends to $\langle f, \phi \rangle$ and this completes the proof.

An immediate consequence of the above inversion theorem is the following uniqueness theorem:

Theorem 4.2 Let $F(\tau) = MF[f]$ and $G(\tau) = MF[g]$ with $f, g \in \mathcal{E}'(I)$ and assume that $F(\tau) = G(\tau)$ for all $\tau > 0$. Then $f = g$.

References


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